

# Lipschitz Continuity and Strong Unicity in G. Freud's Work

HANS-PETER BLATT

*Mathematisch-Geographische Fakultät, Katholische Universität Eichstätt,  
Ostenstrasse 18, D-8078 Eichstätt, West Germany*

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DEDICATED TO THE MEMORY OF GÉZA FREUD

The purpose of this paper is to illustrate the significance of G. Freud's work on Lipschitz continuity of the operator of best uniform approximation and the impact of his investigations on strong unicity and strong unicity constants. © 1986

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Let  $C[a, b]$  be the collection of all continuous real-valued functions on the interval  $[a, b]$  endowed with the uniform norm  $\|\cdot\|$ , and  $V_n$  a subspace of  $C[a, b]$  of dimension  $n$ . For each  $f \in C[a, b]$  let

$$P_{V_n}(f) := \{v \in V_n \mid \|f - v\| = \rho_n(f)\} \quad (1)$$

be the set of all best uniform approximations of  $f$  from  $V_n$ , where

$$\rho_n(f) := \inf_{v \in V_n} \|f - v\|. \quad (2)$$

If  $V_n$  is a Haar subspace on  $[a, b]$ , then there exists for every  $f \in C[a, b]$  exactly one element  $v_f$  in  $P_{V_n}(f)$ , and we define the operator

$$T_n: C[a, b] \rightarrow V_n \quad (3)$$

of best uniform approximation by  $T_n(f) := v_f$ . Moreover, for each  $\varepsilon \geq 0$  define

$$P_{V_n}^f(\varepsilon) := \{v \in V_n \mid \|f - v\| \leq (1 + \varepsilon) \rho_n(f)\}. \quad (4)$$

Then for  $V_n = \prod_{n-1}$ , the collection of all real polynomials of degree  $\leq n-1$ , C. de la Vallée Poussin [16] proved that the set-valued mapping

$$P_{V_n}^f: [0, \infty) \rightarrow 2^{V_n}$$

is continuous at the point 0. Consequently, if  $V_n = \prod_{n-1}$ , the mapping  $T_n$  is continuous at every point  $f \in C[a, b]$ .

Freud [5] seems to have been the first to give a quantitative version of this theorem of de la Vallée Poussin, namely to investigate the pointwise Lipschitz continuity of the operator  $T_n$ .

**THEOREM 1** (Freud [5]). *If  $V_n$  is a Haar subspace of  $C[a, b]$ , then there exists a constant  $A > 0$ ,  $A$  depending on  $f$  and  $V_n$ , such that*

$$\|T_n(f) - T_n(g)\| \leq A \|f - g\| \quad \text{for all } g \in C[a, b]. \quad (5)$$

It is very interesting to look into the original proof of this theorem, especially because nowadays Theorem 1 is usually proved by using the strong unicity theorem (cf. [3]): First note that

$$|\rho_n(f) - \rho_n(g)| \leq \|f - g\|. \quad (6)$$

Moreover, there exist  $n+1$  points  $a \leq x_0 < x_1 < \dots < x_n \leq b$  such that

$$\varepsilon(-1)^i (f - T_n(f))(x_i) = \rho_n(f) \quad (7)$$

( $\varepsilon = +1$  or  $\varepsilon = -1$ ) for  $i = 0, \dots, n$ .

Then, for  $0 \leq i \leq n$  and any  $v \in V_n$ ,

$$\begin{aligned} & \varepsilon(-1)^i (T_n(f) - v)(x_i) \\ &= \varepsilon(-1)^i [(T_n(f) - f)(x_i) + (f - v)(x_i)] \\ &\leq -\rho_n(f) + \|f - v\|. \end{aligned} \quad (8)$$

If  $v = T_n(g)$ ,  $g \in C[a, b]$ , and  $\|f - g\| \leq \delta$ , then the inequalities (6) and (8) yield

$$\varepsilon(-1)^i (T_n(f) - v)(x_i) \leq 2\delta. \quad (9)$$

Omitting the  $r$ th point ( $0 \leq r \leq n$ ), Freud considered now the point set

$$x_0 < x_1 < \dots < x_{r-1} < x_{r+1} < \dots < x_n \quad (10)$$

and the fundamental polynomials  $L_{i,r}(x) \in V_n$ ,  $0 \leq i \leq n$ ,  $i \neq r$ , of Lagrange interpolation corresponding to the set (10):

$$L_{i,r}(x) = \delta_{ij} \quad \text{for } 0 < i, j \leq n; i, j \neq r.$$

Further, let  $x_{-1} := a$ ,  $x_{n+1} := b$  and define

$$\lambda_r := \max_{x_{r-1} \leq x \leq x_{r+1}} \sum_{\substack{i=0 \\ i \neq r}}^n |L_{i,r}(x)| \quad \text{for } 0 \leq r \leq n, \quad (11)$$

$$\lambda_{-1} := \max_{a \leq x \leq x_0} \sum_{i=0}^{n-1} |L_{i,n}(x)|, \quad (12)$$

$$\lambda_{n+1} := \max_{x_n \leq x \leq b} \sum_{i=1}^n |L_{i,0}(x)|, \quad (13)$$

$$\lambda := \max_{-1 \leq i \leq n+1} \lambda_i. \quad (14)$$

Since

$$\operatorname{sgn} L_{i,r}(x) = (-1)^{i-r+1} \quad \text{for } x \in (x_{r-1}, x_{r+1}), \quad (15)$$

we obtain for the function

$$\tilde{v}(x) := \varepsilon(T_n(f) - T_n(g))(x) \in V_n$$

and  $x \in [x_{r-1}, x_{r+1}]$ ,  $0 \leq r \leq n$ ,

$$\begin{aligned} (-1)^{r+1} \tilde{v}(x) &= (-1)^{r+1} \sum_{\substack{i=0 \\ i \neq r}}^n \tilde{v}(x_i) L_{i,r}(x) \\ &= \sum_{\substack{i=0 \\ i \neq r}}^n \varepsilon(-1)^i (T_n(f) - T_n(g))(x_i) |L_{i,r}(x)| \\ &\leq 2\delta \lambda_r. \end{aligned} \quad (16)$$

Combining (16) with the corresponding inequality for the index  $r+1$ , we get  $|\tilde{v}(x)| \leq 2\delta \lambda$  for  $x \in [x_r, x_{r+1}]$  and each  $0 \leq r \leq n-1$  or

$$|\tilde{v}(x)| \leq 2\delta \lambda \quad \text{for } x \in [x_0, x_n]. \quad (17)$$

Since

$$\operatorname{sgn} L_{i,n}(x) = (-1)^i \quad \text{for } x \in [a, x_0], i = 0, \dots, n-1, \quad (18)$$

we get

$$\tilde{v}(x) = \sum_{i=0}^{n-1} \tilde{v}(x_i) L_{i,n}(x) \leq 2\delta \lambda_{-1}; \quad (19)$$

the same arguments lead to

$$\operatorname{sgn} L_{i,0}(x) = (-1)^{n-i} \quad \text{for } x \in (x_n, b], i = 1, \dots, n, \quad (20)$$

and therefore

$$(-1)^n \tilde{v}(x) = (-1)^n \sum_{i=1}^n \tilde{v}(x_i) L_{i,0}(x) \leq 2\delta \lambda_{n+1}. \quad (21)$$

Combining (19) and (21) with the inequality (16) for  $r=0$  and  $r=n$ , we obtain finally together with (17),

$$\|\tilde{v}\| \leq 2\delta \lambda$$

or

$$\|T_n(f) - T_n(g)\| \leq 2\lambda \|f - g\|. \quad \blacksquare \quad (22)$$

Since the arguments in the proof of Freud do not depend on  $T_n(g) \in V_n$ , but may be applied to any  $v \in V_n$ , Freud has actually proved that

$$\|T_n(f) - v\| \leq \lambda \max_{0 \leq i \leq n} [\varepsilon (-1)^i (T_n(f) - v)(x_i)]$$

or, using (8),

$$\|f - v\| \geq \|f - T_n(f)\| + \lambda^{-1} \|T_n(f) - v\|. \quad (23)$$

This means that the best uniform approximation  $T_n(f)$  is *strongly unique* for every  $f \in C[a, b]$ , a result extended by Newman and Shapiro [11] for the approximation of  $f \in C(X)$  by a Chebyshev system, where  $C(X)$  is the Banach space of continuous functions on a compact Hausdorff space  $X$ . Moreover Freud has given a lower bound for the *strong unicity constant* which is the largest constant  $\gamma$ , denoted by  $\gamma_n(f)$ , such that the inequality

$$\|f - v\| \geq \|f - T_n(f)\| + \gamma \|T_n(f) - v\| \quad (24)$$

holds. Namely, from (23), we know that

$$\gamma_n(f) \geq \lambda^{-1}, \quad (25)$$

where  $\lambda$  is defined by (14).

To connect the lower bound (25) with the lower bound given by Cline [4], let us consider again the point set  $a \leq x_0 < x_1 < \dots < x_n \leq b$  satisfying (7). We define the function  $v_r \in V_n$  such that

$$v_r(x_i) = \varepsilon (-1)^i = \operatorname{sgn} (f - T_n(f))(x_i) \quad (26)$$

for  $i = 0, \dots, n$ ,  $i \neq r$ . Then the definitions (11)–(14) and the properties (15), (18), (20) yield

$$K := \max_{0 \leq r \leq n} \|v_r\| \geq \lambda. \quad (27)$$

So  $1/K$  is again a lower bound for the strong unicity constant, a result proved by Cline [4] for the approximation of  $f \in C(X)$  by a Chebyshev system.

At first glance, by (27), the lower bound of Freud in (25) seems to be stronger than the lower bound  $\gamma_n(f) \geq 1/K$  of Cline. But a closer look shows that  $K = \lambda$  and both lower bounds are the same.

In the last ten years quite a number of papers were concerned with estimations for strong unicity constants, especially in the case  $V_n = \prod_{n-1}$ : Poreda [13] raised the question to describe the asymptotic behaviour of  $\gamma_n(f)$  for fixed  $f$  and  $n \rightarrow \infty$ . Later, Henry and Roulier [6] conjectured that

$$\lim_{n \rightarrow \infty} \gamma_n(f) = 0 \quad \text{or} \quad f \text{ is a polynomial} \quad (28)$$

if  $V_n = \prod_{n-1}$ ,  $n = 1, 2, \dots$ . This problem is still open, it is only solved for special classes of functions [2, 6–10, 14]. Finally we want to mention that recently Kroo [10] used the lower bound in (21) for obtaining lower asymptotic estimates for  $\gamma_n(f)$ ,  $n \rightarrow \infty$ .

Another interesting aspect of Freud's proof of Theorem 1 is reflected in the case when the subspace  $V_n$  does not satisfy the Haar condition:

Let  $V_n$  be a weak Chebyshev subspace satisfying the condition (N), i.e., there exists a point  $z \in [a, b]$  such that each  $v \in V_n$ ,  $v \neq 0$ , has at most  $n - 1$  zeros in  $[a, b] \setminus \{z\}$ . Then for every  $f \in C[a, b]$  there exists a best uniform approximation  $v_f \in V_n$ , characterized by an alternating point set

$$a \leq x_0 < x_1 < \dots < x_n \leq b, \quad (29)$$

i.e.,

$$\varepsilon(-1)^i (f - v_f)(x_i) = \rho_n(f) \quad (30)$$

( $\varepsilon = +1$  or  $\varepsilon = -1$ ).  $v_f$  is called *alternation element* of  $P_{V_n}(f)$  and  $v_f$  is unique (Nürnberg, Sommer [12], Sommer, Strauß [15]).

**THEOREM 2.** *Let  $V_n$  be a weak Chebyshev subspace of  $C[a, b]$  with property (N). Then the selection  $s: C[a, b] \rightarrow V_n$ , which associates to every  $f \in C[a, b]$  the unique alternation element  $s(f) = v_f$  of  $P_{V_n}(f)$ , is a pointwise Lipschitz continuous mapping.*

*Proof.* Assume,  $s$  is not Lipschitz continuous at  $f \in C[a, b]$ . Then  $f \notin V_n$  and there exists a sequence  $\{f_m\}$  in  $C[a, b]$  such that  $f_m \rightarrow f$  and

$$\|s(f) - s(f_m)\| > m \|f - f_m\|. \quad (31)$$

$f - s(f_m)$  has an alternating point set

$$a \leq x_0^{(m)} < x_1^{(m)} < \cdots < x_n^{(m)} \leq b, \quad (32)$$

where  $\varepsilon_m (-1)^i (f_m - s(f_m))(x_i^{(m)}) = \rho_n(f_m)$  for  $i = 0, \dots, n$  ( $\varepsilon_m = +1$  or  $\varepsilon_m = -1$ ). We may assume that  $x_i^{(m)} \rightarrow x_i$  for  $m \rightarrow \infty$  ( $i = 0, \dots, n$ ), and  $\varepsilon_m = \varepsilon$  ( $\varepsilon = +1$  or  $\varepsilon = -1$ ) is fixed for all  $m$ . Then

$$a \leq x_0 < x_1 < \cdots < x_n \leq b, \quad (33)$$

where  $\varepsilon (-1)^i (f - s(f))(x_i) = \rho_n(f)$  for  $i = 0, \dots, n$ . Now we omit a fixed point  $x_r$  in (33), where  $x_r$  is chosen as the point  $z$  in the condition (N), if  $z \in \{x_0, x_1, \dots, x_n\}$ . Following Freud's proof of Theorem 1, we obtain, as in (16), for  $x \in [x_{r-1}, x_{r+1}]$  that

$$(-1)^{r+1} \varepsilon (s(f) - s(f_m))(x) \leq 2\lambda_r \|f - f_m\|, \quad (34)$$

where  $x_{-1} := a$ ,  $x_{n+1} := b$  and  $\lambda_r$  is defined by (11). Analogously we have for  $x \in [x_{r-1}^{(m)}, x_{r+1}^{(m)}]$  the inequality

$$(-1)^{r+1} \varepsilon (s(f_m) - s(f))(x) \leq 2\lambda_r^{(m)} \|f - f_m\|, \quad (29)$$

where again  $x_{-1}^{(m)} := a$ ,  $x_{n+1}^{(m)} := b$ ,

$$\lambda_r^{(m)} := \max_{x_{r-1}^{(m)} \leq x \leq x_{r+1}^{(m)}} \sum_{\substack{i=0 \\ i \neq r}}^n |L_{i,r}^{(m)}(x)| \quad (36)$$

and the functions  $L_{i,r}^{(m)}(x) \in V_n$  are the fundamental polynomials of Lagrange interpolation corresponding to the set

$$x_0^{(m)} < x_1^{(m)} < \cdots < x_{r-1}^{(m)} < x_{r+1}^{(m)} < \cdots < x_n^{(m)}.$$

Since  $L_{i,r}^{(m)} \rightarrow L_{i,r}$ , and therefore  $\lambda_r^{(m)} \rightarrow \lambda_r$  for  $m \rightarrow \infty$ , there exists a constant  $\gamma_1 > 0$  such that

$$|(s(f) - s(f_m))(x)| \leq \gamma_1 \|f - f_m\| \quad (37)$$

for all  $x \in [x_{r-1} + \delta, x_{r+1} - \delta]$  with  $\delta := (x_{r+1} - x_{r-1})/4$ . Since for  $v \in V_n$  the uniform norm on  $[a, b]$  is equivalent to the uniform norm on the interval  $[x_{r-1} + \delta, x_{r+1} - \delta]$ , inequality (37) contradicts (31).

A generalization of Theorem 2 for generalized spline spaces  $V_n$  is proved by Blatt, Nürnberger, Sommer [1], again using as an essential tool to the idea of Freud, outlined in the above proof.

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